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Symmetry Breaking, Central Charges and the $\mathrm{AdS}_2/\mathrm{CFT}_1$ Correspondence

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Abstract

When two-dimensional Anti-de Sitter space (AdS_2) is endowed with a non-constant dilaton the origin of the central charge in the Virasoro algebra generating the asymptotic symmetries of AdS_2 can be traced back to the breaking of the $SL(2,R)$ isometry group of AdS_2 . We use this fact to clarify some controversial results appeared in the literature about the value of the central charge in these models.

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The Anti-de Sitter(AdS)/conformal field theory (CFT) correspondence in two spacetime dimensions [1, 2, 3] seems to contradict the general belief that low-dimensional physics is simpler than the higher-dimensional one. Compared with the higher-dimensional cases, the AdS_2/CFT_1 duality has many puzzling and controversial features. Even though general arguments suggest that gravity on AdS_2 should be related to some sort of conformal mechanics living on the one-dimensional boundary of AdS_2 , the realization of this correspondence is rather involved. Although it has been shown that, analogously to the three-dimensional (3D) case [4], the conformal symmetry involved is infinite dimensional [2], the search for a physical system realizing this symmetry has not been very successful. On the other hand, the fact that AdS_2 has a disconnected boundary makes the AdS_2/CFT_1 correspondence problematic even at a more fundamental level.

At a technical level there are features of gravity on AdS_2 that represent a further difficulty for sheding light on the subject, e.g. the existence of several two-dimensional (2D) gravity models admitting AdS_2 as solution (models with or without dilatons) or the fact that the boundary of AdS_2 is one-dimensional.

In a previous work [2] , we have been able, working in the context of 2D dilaton gravity, to show that the asymptotic symmetry group of AdS_2 is generated by a Virasoro algebra. Using a canonical realization of the symmetry we have also computed the central charge of the algebra. Unfortunately, using our value of the central charge for the computation of the statistical entropy of 2D black holes, we found a discrepancy of a factor of $\sqrt{2}$ with respect to the thermodynamical entropy.

Later Navarro-Salas and Navarro [3], using a boundary field realization of the symmetry, found a value of the central charge (half of our result) that produces a statistical entropy in agreement with the thermodynamical one.

In this letter we try to clarify this controversial point. We show that in models where AdS_2 is endowed with a non-constant dilaton, the origin of the central charge of the Virasoro algebra can be traced back to the breakdown of the $SL(2,R)$ isometry group of AdS_2 . The authors of Ref. [3] use instead a non-scalar dilaton, which enables them to keep the $SL(2,R)$ symmetry of AdS_2 unbroken. However, using a non-scalar dilaton, they have to give up the diffeomorphisms invariance of the 2D dilaton gravity theory. We will show that keeping the dilaton to transform as a scalar, hence considering a truly diffeomorphisms invariance 2D dilaton gravity theory, our previous result of Ref. [2] for the central charge can be recovered also adopting a boundary field realization of the asymptotic symmetry of AdS_2 .

Let us consider a generic two-dimensional (2D) dilaton gravity model,

$$S = \frac{1}{2}\int d^2x\, \sqrt{-g}\, \left[\eta R + \lambda^2 V(\eta)\right], \tag{1}$$

where η is a scalar field related to the usual definition of the dilaton ϕ by $\eta = \exp(-2\phi)$. In the discussion of the symmetries of the model a crucial role is played by the scalar η. In general, the scalar character of η implies that the symmetries (isometries) of the 2D spacetime are broken by a non-costant dilaton. In fact under the isometric transformations generated by a Killing vector χ,

$$\delta\eta = \mathcal{L}_\chi \eta = \chi^\mu \partial_\mu \eta. \tag{2}$$

A non-constant dilaton in general implies $\mathcal{L}_\chi \eta \neq 0$. On the other hand, 2D dilaton gravity always admits a Killing vector given by [5]:

$$\chi^\mu_{(1)} = \epsilon^{\mu\nu}\partial_\nu \eta, \tag{3}$$

which leaves η invariant. Thus, a non-constant field η in general breaks down the isometry group of the metric to the subgroup generated by the Killing vector $\chi_{(1)}$.

The previous features of 2D dilaton gravity theories have strong analogies with spontaneous symmetry breaking in ordinary field theory. In the case under consideration the quantity characterizing the breaking of the symmetry is $\partial_\mu \eta$. The analogy is particularly evident if one considers classical solutions of the model (1). These solutions are characterized by a mass M and a temperature T (when the solutions can be interpreted as black holes). Both M and T are geometric objects, so that they are invariant under the isometry group of the metric, and can be expressed in terms of η [5, 6]

$$M = F_0 \left[\int d\eta \lambda^2 V(\eta) - (\nabla \eta)^2 \right], \qquad T \propto V(\eta_h), \tag{4}$$

where F_0 is a constant related to the normalization of the Killing vectors (in the following we use $F_0 = 1/2\lambda\eta_0$) and η_h is the scalar evaluated on the horizon of the 2D black hole. Using the field equations one can easily show that the solutions characterized by a constant dilaton, thus preserving the isometry group of the metric, have zero mass and temperature. Only symmetry-breaking excitations, characterized by a non constant dilaton, can have $M \neq 0$, $T \neq 0$.

Let us now apply the previous considerations to AdS$_2$. With a dilaton potential $V = 2\eta$ the action (1) describes the Jackiw-Teitelboim (JT) model [7]. AdS$_2$ or more generally black holes in AdS$_2$, are solutions of the model [8]. Owing to Birkhoff's theorem, the solutions can always be written in a static form, with dilaton $\eta = \eta_0 \lambda x$, where η_0 is an integration constant (the field equations, but not the action, are invariant for rescaling of η by a constant) and metric

$$ds^2 = -(\lambda^2 x^2 - a^2)dt^2 + (\lambda^2 x^2 - a^2)^{-1} dx^2, \tag{5}$$

where a^2 is proportional to the mass of the black hole.

Being a maximally symmetric spacetime AdS$_2$ admits three Killing vectors generating the $SO(1,2) \sim SL(2,R)$ group of isometries. It is evident that the static solution for η is invariant only under the action of the Killing vector (3), which in this case describes time-translations T. Thus, the non-constant value of the dilaton breaks $SL(2,R) \to T$. The parameter characterizing the symmetry breaking is $\partial_x \eta = \eta_0 \lambda$.

Similar considerations hold when one considers the asymptotic symmetries of AdS$_2$. These are defined as the transformations which leave the asymptotic form of the AdS$_2$ metric invariant, and where shown in [2] to generate a Virasoro algebra with non-trivial central charge. Our aim here is to give a realization of this algebra in terms of fields which describe the degrees of freedom of the boundary. For this purpose, it is useful to adopt the formalism introduced in [3].

We define a two-dimensional metric to be asymptotically AdS$_2$ if, for $x \to \infty$, it behaves as

$$g_{tt} = -\lambda^2 x^2 + \gamma_{tt}(t) + o\left(\frac{1}{x^2}\right),$$

$$
\begin{aligned}
g_{tx} &= \frac{\gamma_{tx}(t)}{\lambda^3 x^3} + o\left(\frac{1}{x^5}\right) , \\
g_{xx} &= \frac{1}{\lambda^2 x^2} + \frac{\gamma_{xx}(t)}{\lambda^4 x^4} + o\left(\frac{1}{x^6}\right) ,
\end{aligned}
\tag{6}
$$

where the fields $\gamma_{\mu\nu}$ parametrize the first sub-leading terms in the expansion and can be interpreted as deformations on the boundary.

The asymptotic form (6) of the metric is preserved by infinitesimal diffeomorphisms $\chi^\mu(x,t)$ of the form [2]

$$
\begin{aligned}
\chi^t &= \epsilon(t) + \frac{\ddot{\epsilon}(t)}{2\lambda^4 x^2} + \frac{\alpha^t(t)}{x^4} + o\left(\frac{1}{x^5}\right) , \\
\chi^x &= -x\dot{\epsilon}(t) + \frac{\alpha^x(t)}{x} + o\left(\frac{1}{x^2}\right) .
\end{aligned}
\tag{7}
$$

where $\epsilon(t)$ and $\alpha^\nu(t)$ are arbitrary and a dot denotes time derivative. α^ν describes "pure gauge" 2D diffeomorphisms which affect only the fields on the boundary. In Ref. [2] is shown that the symmetries (7) are generated by a Virasoro algebra.

In view of (2), the asymptotic behaviour of the scalar field η, compatible with the transformations (7), must take the form

$$
\eta = \eta_0 \left(\lambda\rho(t)\, x + \frac{\gamma_{\phi\phi}(t)}{2\lambda x}\right) + o\left(\frac{1}{x^3}\right) , \tag{8}
$$

where ρ and $\gamma_{\phi\phi}$ play a role analogous to that of the $\gamma_{\mu\nu}$. On shell hold the constraints

$$
\lambda^{-2}\ddot{\rho} = \rho(\gamma_{tt} - \gamma_{xx}) - \gamma_{\phi\phi}, \tag{9}
$$

$$
\dot{\rho}\gamma_{tt} + \frac{\rho}{2}\dot{\gamma}_{xx} + \dot{\gamma}_{\phi\phi} = 0. \tag{10}
$$

Eq. (8) implies that the full, infinite dimensional, asymptotic symmetry of AdS_2 is broken by the boundary condition for η. In fact, only the asymptotic transformations generated by the Killing vector (3), which correspond to $\epsilon = \rho$ in Eqs. (7), leave η asymptotically invariant. The quantity characterizing the symmetry breaking is now the boundary field $\rho = \partial_x \eta + o(x^{-1})$.

In [3] it was assumed that $\rho(t) \equiv 1$. However, this is at variance with the transformation (2) of η under diffeomorphisms, which implies that the background dilaton $\eta = \eta_0 \lambda x$, is transformed by (7) into the form (8) with generic $\rho(t)$. In order to avoid this problem, the authors of [3] had instead to assume that under diffeomorphisms the dilaton transform as $\delta\eta = \chi^\mu \partial_\mu \eta + \partial_t \chi^t \eta$ [9], but this seems rather ad hoc and moreover spoils the diffeomorphism invariance of the action (1), presumably leading to inconsistencies. It is important to notice that, if η transforms according to the previous transformation law, the asymptotic symmetry group of AdS_2 is no longer broken. In fact, in this case, asymptotically $\delta\eta = 0$ under the full group generated by the Killing vectors (7).

Although the asymptotic symmetry of AdS_2 is broken by η, the boundary fields γ_{tt}, γ_{xx}, $\gamma_{\phi\phi}$, ρ still span a representation of the full infinite dimensional group generated by

the Killing vectors (7). In fact, under the asymptotic symmetries (7), the boundary fields transform as

$$\begin{aligned}
\delta\gamma_{tt} &= \epsilon\dot{\gamma}_{tt} + 2\dot{\epsilon}\gamma_{tt} - \frac{\dddot{\epsilon}}{\lambda^2} - 2\lambda^2\alpha^x, \\
\delta\gamma_{xx} &= \epsilon\dot{\gamma}_{xx} + 2\dot{\epsilon}\gamma_{xx} - 4\lambda^2\alpha^x, \\
\delta\gamma_{\phi\phi} &= \epsilon\dot{\gamma}_{\phi\phi} + \dot{\epsilon}\gamma_{\phi\phi} + \frac{\ddot{\epsilon}\dot{\rho}}{\lambda^2} + 2\lambda^2\rho\alpha^x, \\
\delta\rho &= \epsilon\dot{\rho} - \dot{\epsilon}\rho.
\end{aligned} \tag{11}$$

These transformations are easily recognized as (anomalous) transformation laws for conformal fields of weight respectively $2, 2, 1, -1$. The anomalous parts of the transformation (the terms proportional to $\ddot{\epsilon}$ and $\dddot{\epsilon}$) are related with the $SL(2, R)$ symmetry breaking. In fact, the anomalous terms are connected one with the other by using the equation of motion (9), whereas the anomalous term appearing in the transformation of $\gamma_{\phi\phi}$ is proportional to $\dot{\rho}$. It follows that, when AdS_2 is endowed with a non-constant dilaton, the origin of the anomalous terms in the transformation laws for conformal boundary fields (hence of the central charge) can be traced back to the breakdown of the full asymptotic isometry group of the spacetime. This implies that in general one has a ρ-dependent central charge. However, we are only interested in classical solutions of the 2D gravity model. Because of Birkhoff's theorem, we can, without loss of generality, limit ourselves to the field configurations with $\rho =$ const. Moreover, the equations of motions of the JT model are invariant under the rescaling of η by a constant. It will therefore be sufficient to consider only the $\rho = 1$ configuration.

The next step in our analysis is the construction of the generator of the conformal symmetry in terms of the boundary fields $\gamma_{tt}, \gamma_{xx}, \gamma_{\phi\phi}, \rho$. This generator has a natural interpretation as the stress-energy tensor associated with the one-dimensional conformal field theory living on the boundary of AdS_2 . A natural candidate for such a generator is the charge $J(\epsilon)$, which in the canonical formalism can be associated with the asymptotic symmetries (7). The charge $J(\epsilon)$ is defined in terms of the boundary contribution one must add to the Hamiltonian in order to have well defined variational derivatives [2],

$$\delta J = -\lim_{x\to\infty}[N(\sigma^{-1}\delta\eta' - \sigma^{-2}\eta'\delta\sigma) - N'(\sigma^{-1}\delta\eta) + N^x(\Pi_\eta\delta\eta - \sigma\delta\Pi_\sigma)]. \tag{12}$$

Using the boundary conditions (6), (8) one finds,

$$\delta J[\epsilon] = \eta_0\left[\lambda\epsilon(\gamma_{tt}\delta\rho + \frac{\rho}{2}\delta\gamma_{xx} + \delta\gamma_{\phi\phi}) + \frac{\dot{\epsilon}\delta\dot{\rho}}{\lambda} - \frac{\ddot{\epsilon}\delta\rho}{\lambda}\right]. \tag{13}$$

This expression is locally integrable in a neighborhood of the classical solutions [2], but unfortunately is not integrable globally in the full phase space. The non-integrability of δJ is, again, a consequence of the $SL(2, R)$ symmetry breaking. For instance, if one uses the symmetry-preserving boundary conditions for η used in Ref. [3], δJ is globally integrable and Eq. (13) yields the form of J obtained in that paper. Fortunately, we do not need to know J globally, but is sufficient to know the form of J near the $\rho = 1$ configuration. The expression of J can be further simplified considering only on-shell field configurations. Using the equations of motion (9) and (10), and expanding around the classical solutions, $\rho = 1 + \bar{\rho}$ in Eq. (13), one obtains at leading order in $\bar{\rho}$,

$$J(\epsilon) = \frac{\eta_0}{\lambda}\left(\dot{\epsilon}\dot{\bar{\rho}} - \ddot{\epsilon}\bar{\rho}\right) + \epsilon M, \tag{14}$$

Where M is the mass (4), which on shell becomes constant. The charge (14) is defined up to an additive constant, we use a normalization such that $J(\epsilon = 1) = M$. In the following we will consider, for sake of semplicity, only the case $M = 0$, i.e variations near the ground state. We are mainly interested in the value of the central charge of the Virasoro algebra, which is independent of M.

For generic ϵ the charges $J(\epsilon)$ are not conserved. The only conserved charge is obtained for $\epsilon = 1$. This fact is a consequence of the $SL(2,R) \to T$ symmetry breaking: only the charge associated with the residual symmetry T is conserved. This fact makes it impossible to use the charges $J(\epsilon)$ to give a realization of the Virasoro algebra, as it has been already shown in the canonical framework [2]. To solve the problem, we proposed to introduce the time-integrated charges,

$$\hat{J}(\epsilon) = \frac{\lambda}{2\pi}\int_0^{\frac{2\pi}{\lambda}} dt\, J(\epsilon). \tag{15}$$

The charges $\hat{J}$ are now trivially conserved and generate the asymptotic symmetries of AdS_2 trough the relation [2]

$$\delta_\omega \widehat{J(\epsilon)} = \left[\hat{J}(\epsilon), \hat{J}(\omega)\right], \tag{16}$$

where the hat means overall time-integration as defined in Eq. (15).

The definitions (15) imply that J is defined up to a total time-derivative. Using this freedom, we can always write,

$$J(\epsilon) = -2\frac{\eta_0}{\lambda}\epsilon\ddot{\bar{\rho}} = \epsilon\Theta_{tt}. \tag{17}$$

Using the transformation laws (11), one gets

$$\epsilon\delta_\omega\Theta_{tt} = \epsilon\left(\omega\dot{\Theta}_{tt} + 2\dot{\omega}\Theta_{tt}\right) + c(\epsilon,\omega), \tag{18}$$

where $c(\epsilon,\omega) = \frac{\eta_0}{\lambda}(\ddot{\epsilon}\,\dot{\omega} - \ddot{\omega}\,\dot{\epsilon})$.

The previous equation tells us that Θ_{tt} can be considered as the one-dimensional stress-energy tensor associated with the conformal symmetry, whereas $\hat{J}(\epsilon)$ are the charges generating a central extension of the Virasoro algebra. Notice that the central charge c in general is ρ-dependent, the expression quoted above being the central charge evaluated near $\rho = 1$. Expanding in Fourier modes,

$$\hat{J}(\epsilon) = \sum_m a_m L_m, \qquad \epsilon = \sum_m a_m e^{i\lambda m t}, \qquad \Theta_{tt} = \sum_m L_m e^{-i\lambda m t}, \tag{19}$$

and using Eqs. (16) and (18) one finds that the L_m span a Virasoro algebra,

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{C}{12}m^3\delta_{m+n}, \tag{20}$$

with central charge $C = 24\eta_0$. The value of the central charge of the algebra is exactly the same as that found by us using a canonical realization of the symmetry [2] and differs by a factor 1/2 from that found by Navarro-Salas and Navarro [3]. The origin of the mismatch

is easily understood. As already pointed out, the authors of Ref. [3] use different boundary conditions and a different transformation law for the field η. They use a non-scalar dilaton and $\rho = 1$, identically, so that the $SL(2, R)$ isometry of AdS_2 is not broken, and the origin of the central charge is completely different from our case. In Ref. [3] the central charge arises as consequence of the anomalous term in the transformation law of γ_{tt} (see Eq. (11)). This is very similar to the three-dimensional case [4]. Their results hold only for a 2D dilaton gravity model that is not invariant under space-time diffeomorphisms. Therefore, the mismatch between statistical and thermodynamical entropy of 2D black hole discussed in our previous work [2] still remains an open question.

References

[1] A. Strominger, J. High Energy Phys. **01** (1999) 007; J. Michelson and A. Strominger J. High Energy Phys. **9909** (1999) 005; M. Spradlin and A. Strominger, J. High Energy Phys. **9911** (1999) 021; G. W. Gibbons and P.K. Townsend, Phys. Lett. **B454** (1999) 187; J. Maldacena, J. Michelson and A. Strominger, J. High Energy Phys. **9902** (1999) 011; S. Cacciatori, D. Klemm and D. Zanon, `hep-th/9910065` ; E. Deotto, G. Furlan, E. Gozzi, `hep-th/9910220`.

[2] M. Cadoni and S. Mignemi, Phys. Rev. **D59** (1999) 081501; Nucl. Phys. **B557** (1999) 165.

[3] J. Navarro-Salas and P. Navarro, `hep-th/ 9910076`.

[4] J. D. Brown and M. Henneaux, Commun. Math. Phys. **104**, 207 (1986); A. Strominger, J. High Energy Phys. **02** (1998) 009; S. Carlip, Phys. Rev. **D51**, 632 (1995); Class. Quant. Grav. **15**, 3609 (1998).

[5] R.B. Mann Phys. Rev. **D47**, 4438 (1993).

[6] M. Cadoni, Phys. Rev. **D53**, 4420 (1996); Phys. Lett. **B395**, 10 (1997).

[7] C. Teitelboim, in "Quantum theory of Gravity" edited by S.M. Christensen (Hilger, Bristol); R. Jackiw, ibid.

[8] M. Cadoni and S. Mignemi Phys. Rev. **D51** (1995) 4319.

[9] J. Navarro-Salas, private communication.